Synthesizing DNFs Using Boolean Algebra

The procedure presented in class for synthesizing disjunctive normal forms involved these steps:

1. Use the *DeMorgan*, $\overline{x\,y} \rightarrow \overline{x} + \overline{y}$ and $\overline{x + y} \rightarrow \overline{x}\,\overline{y}$ repeatedly to distribute all negations down to variables. Use the *negation* identity, $\overline{\overline{x}} \rightarrow x$ to simplify the resulting literals.

2. Use *distributivity,* $x\,(y+z) \rightarrow x\,y+x\,z$ repeatedly to move all the products inside of the sums.

3. If you repeat Steps 1 and 2 until they no longer apply, you have synthesized an equivalent *sum-of-products* (*SOP*) form, but it is not normalized because some product clauses may not include all the variables contained in the original expression. Use the identity $X \rightarrow Xy + X\overline{y}$ as necessary to introduce variable y to clause X. Repeat as needed to introduce all the variables to every clause.

4. Use *idempotence*, $x\,x \rightarrow x$ and $x + x \rightarrow x$ to eliminate redundant terms.

Along the way, use the absorption laws, $x + xy \rightarrow x$ and $x\,(x + y) \rightarrow x$ and other identities to simplify until you reach Step 3.

EXAMPLE. *Use boolean algebra to synthesize a DNF for* $\overline{p\,q}(r + \overline{p + s})$.

$$
\begin{aligned}
& \overline{p\,q}(r + \overline{(p + s)}) \\
&= (\overline{p} + \overline{q})\,(r + \overline{p}\,\overline{s}) && \text{(DeMorgan } \times 2\text{)} \\
&= (\overline{p} + \overline{q})\,r + (\overline{p} + \overline{q})\,\overline{p}\,\overline{s} && \text{(distributivity)} \\
&= \overline{p}\,r + \overline{q}\,r + \cancel{\overline{p}}\,\overline{p}\,\overline{s} + \overline{p}\,\overline{q}\,\overline{s} && \text{(distributivity } \times 2\text{)} \\
&= \overline{p}\,\overline{q}\,r\,\overline{s} + \overline{p}\,\overline{q}\,r\,s + \overline{p}\,q\,r\,\overline{s} + \overline{p}\,q\,r\,s && (x \rightarrow xy + x\overline{y} \ \times 7) \\
&\quad + \cancel{\overline{p}\,\overline{q}\,r\,\overline{s}} + \cancel{\overline{p}\,\overline{q}\,r\,s} + p\,\overline{q}\,r\,\overline{s} + p\,\overline{q}\,r\,s \\
&\quad + \overline{p}\,\overline{q}\,\overline{r}\,\overline{s} + \cancel{\overline{p}\,\overline{q}\,r\,\overline{s}} + \overline{p}\,q\,\overline{r}\,\overline{s} + \cancel{\overline{p}\,q\,r\,\overline{s}} \\
&\quad + \cancel{\overline{p}\,\overline{q}\,\overline{r}\,\overline{s}} + \cancel{\overline{p}\,\overline{q}\,r\,\overline{s}} \\
&= \overline{p}\,\overline{q}\,\overline{r}\,\overline{s} + \overline{p}\,\overline{q}\,r\,\overline{s} + \overline{p}\,\overline{q}\,r\,s + \overline{p}\,q\,\overline{r}\,\overline{s} + \overline{p}\,q\,r\,\overline{s} + \overline{p}\,q\,r\,s + p\,\overline{q}\,r\,\overline{s} + p\,\overline{q}\,r\,s \\
& && \text{(idempotence, rearranging)}
\end{aligned}
$$