ASYMPTOTICALLY FLAT SELF-DUAL SOLUTIONS TO EUCLIDEAN GRAVITY

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In an attempt to find gravitational analogs of Yang–Mills pseudoparticles, we obtain two classes of self-dual solutions to the euclidean Einstein equations. These metrics are free from singularities and approach a flat metric at infinity.

The discovery of pseudoparticle solutions to the euclidean SU(2) Yang–Mills theory [1] has suggested the possibility that analogous solutions might occur in Einstein's theory of gravitation. The existence of such solutions would have a profound effect on the quantum theory of gravitation [2,3]. Since the Yang–Mills pseudoparticles possess self-dual field strengths, one likely possibility is that gravitational pseudoparticles are characterized by self-dual curvature.

In fact it has been pointed out by Hawking [3] that the Taub-NUT metric [4], when appropriately continued to euclidean space–time, produces a self-dual curvature and hence is a possible candidate for a gravitational pseudoparticle. He has also given a generalized multi-Taub-NUT metric. However, these metrics do not approach a flat metric at infinity [5]. To see this, let us write the euclidean Taub-NUT solution as

$$(\mathrm{d}s)^2 = [(R+m)/(R-m)]\ \mathrm{d}R^2 + 4(R^2-m^2)\{\sigma_x^2 + \sigma_y^2 + (2m/(R+m))^2\sigma_z^2\}, \tag{1}$$

where σ_x, σ_y, σ_z form a standard Cartan basis,

$$\begin{aligned} \sigma_x &= \tfrac{1}{2}(-\cos\psi\ \mathrm{d}\theta - \sin\theta\sin\psi\ \mathrm{d}\phi), \\ \sigma_y &= \tfrac{1}{2}(\sin\psi\ \mathrm{d}\theta - \sin\theta\cos\psi\ \mathrm{d}\phi), \\ \sigma_z &= \tfrac{1}{2}(-\mathrm{d}\psi - \cos\theta\ \mathrm{d}\phi), \end{aligned} \tag{2}$$

obeying the structure equations of the exterior algebra [6],

$$\mathrm{d}\sigma_x = 2\sigma_y \wedge \sigma_z, \tag{3}$$

etc. Here θ, ψ and ϕ are Euler angles on S^3 with ranges $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant \psi \leqslant 4\pi$. Then it is easy to see that the above metric describes a distorted 3-dimensional hypersphere S^3 for any fixed value of $R > m$.

Since a Yang–Mills pseudoparticle approaches a pure gauge at infinity and is interpreted as inducing transitions between topologically inequivalent vacua, one might require that gravitational analogs have a similar asymptotic behavior. In this letter we explore the possibility of gravitational pseudoparticles which possess a self-dual curvature and approach a flat metric at infinity. In the following we present two classes of such solutions. They are both singularity-free in the entire spacetime and their manifolds have a simple topological structure.

In deriving these solutions we exploit a particularly useful choice of gauge (local Lorentz frame). First we define a local orthonormal frame using the vierbeins $e^a{}_\mu$, and take

$$e^a = e^a{}_\mu\ \mathrm{d}x^\mu. \tag{4}$$

In terms of the e^a, the metric is expressed as $\mathrm{d}s^2 = (e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2$. Then the connection one-form $\omega^a{}_b$ is defined by

$$\mathrm{d}e^a = -\omega^a{}_b \wedge e^b, \quad \omega^a{}_b = -\omega^b{}_a. \tag{5}$$

Latin indices are raised and lowered by a flat metric. Then we define the curvature two-form by

$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{6}$$

Now we note that if $\omega^a{}_b$ is self-dual,

$$\omega^0{}_1 = -\omega^2{}_3, \tag{7}$$

etc., then $R^a{}_b$ is self-dual. This follows directly from the definition (6) of $R^a{}_b$. Since any self-dual curvature gives a vanishing Ricci tensor, any metric yielding a self-dual connection is a solution to the Einstein equation. On the other hand, it is easy to show that any self-dual curvature can be obtained, by a suitable change of gauge, from a metric yielding a self-dual connection ‡1. In this "self-dual gauge", the problem of finding a self-dual solution to the Einstein equation [7] is therefore reduced to one of finding self-dual connections and hence solving first-order differential equations generated by eq. (5). This is quite analogous to the Yang–Mills case [1].

In the following we consider two types of metrics having axial symmetry as in the Taub-NUT case ‡2:

$$\mathrm{I:}\ (\mathrm{d}s)^2 = f^2(r)\,\mathrm{d}r^2 + r^2 g^2(r)(\sigma_x^2 + \sigma_y^2) + r^2\sigma_z^2, \tag{8}$$

$$\mathrm{II:}\ (\mathrm{d}s)^2 = f^2(r)\,\mathrm{d}r^2 + r^2(\sigma_x^2 + \sigma_y^2) + r^2 g^2(r)\sigma_z^2. \tag{9}$$

Here we consider these metrics directly in the euclidean space and do not regard them as a result of some continuation from the Minkowski regime. Asymptotic flatness requires that

$$\lim_{r\to\infty} f(r) = \lim_{r\to\infty} g(r) = 1. \tag{10}$$

Taking as our orthonormal frames

$$\mathrm{I:}\ e^a = (f(r)\,\mathrm{d}r, rg(r)\sigma_x, rg(r)\sigma_y, r\sigma_z), \tag{11}$$

$$\mathrm{II:}\ e^a = (f(r)\,\mathrm{d}r, r\sigma_x, r\sigma_y, rg(r)\sigma_z), \tag{12}$$

we find after some simple algebra that the self-duality of the connection implies

$$\mathrm{I:}\ g^2 = f(2g^2 - 1), \qquad f = g(g + rg'), \tag{13}$$

$$\mathrm{II:}\ fg = 1, \quad f(2 - g^2) = g + rg'. \tag{14}$$

Asymptotically flat solutions are given, respectively, by

$$\mathrm{I:}\ f(r) = \tfrac{1}{2}(1 + [1 - (a/r)^4]^{-1/2}), \tag{15}$$

$$g(r) = \{\tfrac{1}{2}(1 + [1 - (a/r)^4]^{1/2})\}^{1/2}, \tag{16}$$

$$\mathrm{II:}\ g(r) = f^{-1}(r) = [1 - (a/r)^4]^{1/2}, \tag{17}$$

where a is an integration constant. The curvature components of case II are given by

$$R^0{}_1 = -R^2{}_3 = -(2a^4/r^6)(e^0 \wedge e^1 - e^2 \wedge e^3),$$
$$R^0{}_2 = -R^3{}_1 = -(2a^4/r^6)(e^0 \wedge e^2 - e^3 \wedge e^1), \tag{18}$$
$$R^0{}_3 = -R^1{}_2 = +(4a^4/r^6)(e^0 \wedge e^3 - e^1 \wedge e^2).$$

The curvatures for case I have the same algebraic form with the replacement

$$2a^4/r^6 \to -a^4/2r^6 g^6. \tag{19}$$

Hence in both cases the curvatures are regular everywhere for $r \geqslant a$ and fall off like $1/r^6$ at infinity. For comparison, we note that the Taub-NUT curvature produced by eq. (1) is obtained by the replacement

$$2a^4/r^6 \to m/(R + m)^3, \tag{20}$$

and thus goes like $1/R^3$ at infinity.

The manifolds described by the above metrics have the topology $\mathrm{R} \times \mathrm{S}^3$. Although the metrics have an apparent singularity at $r = a$, it can be eliminated by a change of variable,

$$u^2 = r^2(1 - (a/r)^4). \tag{21}$$

For instance the solution II now takes the form

$$(\mathrm{d}s)^2 = \mathrm{d}u^2/(1 + (a/r)^4)^2 + u^2\sigma_z^2 + r^2(\sigma_x^2 + \sigma_y^2). \tag{22}$$

Our next task is to compute topological invariants of the manifold. Here, as in the Taub-NUT case [8], we have to be careful about possible contributions from the boundary of the manifold.

$\hat{A}$-genus (axial anomaly). The Atiyah–Patodi–Singer theorem [9] gives the $\hat{A}$-genus of the manifold $[r_1, r_2] \times \mathrm{S}^3$ as

‡1 The proof involves decomposing any given spin connection $\omega^a{}_b$ into self-dual and anti-self-dual parts. If $R^a{}_b$ is self-dual, the anti-self-dual part of $\omega^a{}_b$ is a pure O(4) gauge transformation, $\Lambda^a{}_c(\mathrm{d}\Lambda^{-1})^c{}_b$, and can be gauged away.

‡2 The spherically symmetric ansatz, $\mathrm{d}s^2 = f^2\mathrm{d}r^2 + r^2g^2(\sigma_x^2 + \sigma_y^2 + \sigma_z^2)$, leads to a trivially flat metric when we impose self-duality.

$$\hat{A}(r_1, r_2) = \hat{A}_{\text{vol}} - (\hat{A}_{\text{surf}} + \tfrac{1}{2}(h_D + \eta_D))|_{r_1}^{r_2}. \tag{23}$$

$\hat{A}_{\text{vol}}$ is the volume integral of the Riemann curvature tensor contracted with its dual and $\hat{A}_{\text{surf}}$ gives the contribution due to the deviation of the metric from a product metric on the boundary [10]. h_D is the number of harmonic spinors of the Dirac operator restricted to the boundary and η_D gives its spectral asymmetry [9,11]. Using the formulas in refs. [8] and [11] we obtain

$$\hat{A}(r_1 = a, r_2 = \infty) = \tfrac{1}{4} - 0 + (-\tfrac{1}{6} - \tfrac{1}{12}) = 0, \tag{24}$$

for both solutions I and II. Thus these solutions by themselves will not induce chiral symmetry breakdown, just as in the Taub-NUT case [8].

Euler–Poincaré characteristic (trace anomaly). The Euler–Poincaré characteristic χ is related to the thermal effects of gravitational pseudoparticles [3,12]. To calculate χ, we apply the Chern–Gauss–Bonnet theorem [13],

$$\chi = \chi_{\text{vol}} - \chi_{\text{surf}}|_{r_1}^{r_2}, \tag{25}$$

where χ_{vol} and χ_{surf} are the analogs of $\hat{A}_{\text{vol}}$ and $\hat{A}_{\text{surf}}$ in eq. (23). Using the known formulas, we find for both solutions I and II the Euler characteristic [‡3]

$$\chi(r_1 = a, r_2 = \infty) = 3 - (-1) + (-4) = 0. \tag{26}$$

This of course agrees with the combinatorial calculation for $R \times S^3$.

We observe that at large r, our curvatures fall like $1/r^6$; in contrast, the euclidean Taub-NUT and Schwarzschild solutions fall like $1/r^3$. This suggests that our metrics describe gravitational "dipoles" while Taub-NUT and Schwarzschild describe monopoles. This is probably a sign that our euclidean solutions will not have a meaningful continuation to Minkowski space, as is the case for the Yang–Mills pseudoparticle.

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‡3 It appears that the manifold of solution II can be compactified by adding an S^2 at $r = a$. In this case (see eq. (22)) the manifold acquires the local topology of $D^2 \times S^2$; since as $r \to a$, the D^2 shrinks to a point, the manifold is homotopic to S^2. If we then omit the $r = a$ boundary term in eq. (26), we obtain $\chi = 4$. However, we know $\chi = 2$ for a manifold homotopic to S^2. Hence the Chern–Gauss–Bonnet theorem requires a "corner" correction in this case. A similar situation occurs if one puts a metric on a cone and tries to compute the Euler characteristic using the Gauss–Bonnet theorem without correcting for the apex. For solution I, analogous arguments indicate that the manifold compactified at $r = a$ is homotopic to the manifold of SO(3). Then the apparent Euler characteristic is 4, while the true value is $\chi = 0$. The compactified manifolds admit a spin structure because the second Stiefel–Whitney classes vanish [14]. However, in practice the "corners" may make it difficult to treat the Dirac operator on the whole manifold. If such an operator can be defined, the $\hat{A}$-genus (axial anomaly) would also require "corner" corrections. This problem is under study.

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