Erratum: Geometry of discrete quantum computing

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Section 5.4 of our paper [1] requires a clarification and a correction.

Clarification: Unentangled vs. product states. In conventional quantum mechanics, using the field of complex numbers, a state is unentangled when it can be expressed as a product state or, equivalently, when equation (27) reports its purity to be 1 [2, 3]. When using finite Galois fields $\mathbb{F}_{p^2}$, for particular choices of p, it is possible for equation (27) to produce a purity of 1 for some entangled states. For example, the normalized entangled state $|\Psi\rangle = |011\rangle + (2+\mathrm{i})\,|100\rangle + |101\rangle + |110\rangle$ has $P_{\mathfrak{h}} = 1$ for $p = 7$. In addition, the process of determining whether a *given* state $|\Phi\rangle$ is a product state may depend on p.

Thus, in finite fields, the simplest way to calculate the number of unentangled states is to disregard equation (27) and count the number of product states. This is exactly how the counting in section 5.4 was done, but the paper did not point out that counting relying only on equation (27) might not lead to the same result.

Correction: Maximally entangled states. We present below a new version of section 5.4 that correctly counts maximally entangled states. The rest of the article is independent of this revision.

5.4. Maximal entanglement

Equation (27) for $P_{\mathfrak{h}}$ includes a normalization factor $\frac{1}{n}$. In the discrete case, this normalization factor is undefined when $p \mid n$. Equation (27) also includes a summation of n terms. In the discrete case, certainly when $p \mid n$ but also in other cases, this summation may vanish in the field even if the individual summands are non-zero. These anomalies are irrelevant for the classification of unentangled states as this computation is performed by directly checking the possibility of direct decomposition into product states, disregarding equation (27).

For maximally entangled states, the purity calculation in conventional quantum mechanics using equation (27) produces 0. Given the above observations, in a discrete field, equation (27) may be undefined or may report a purity of 0 even for partially entangled states. For example, the normalized 5-qubit state $|\Psi\rangle = (1-\mathrm{i})\,(|00\rangle + |11\rangle) \otimes |000\rangle$ has $P_{\mathfrak{h}} = 0$ for $p = 3$, and is not maximally entangled because only the first two qubits are entangled. In the discrete case, we therefore check for maximally entangled states using the following equations,

$$\forall j, \forall \mu, \langle \sigma^j_\mu \rangle^2 = 0 \ , \qquad (40)$$

which avoids the normalization factor and simply checks that each summand is 0.

We now implement these procedures to enumerate the maximally entangled states for the specific cases for $n = 2, 3$ and compare these to the counts for product states. We can verify explicitly that unit-norm product states for $n = 2$, $p = \{3, 7, 11, 19, \ldots\}$ occur with frequency

$$(p+1)p^2(p-1)^2 = \{144, 14\,112, 145\,200, 2339\,280, \ldots\} \ ,$$

and for general n,

$$(p+1)p^n(p-1)^n \ .$$

The irreducible state counts are reduced by $(p+1)$, giving

$$p^2(p-1)^2 = \{36, 1764, 12\,100, 116\,964, \ldots\} \ ,$$

and in general for n-qubits, there are $p^n\,(p-1)^n$ instances of pure product states.

Performing the computation using equation (40), we find maximally entangled states with frequencies for two qubits of

$$p\left(p^2-1\right)(p+1) = \{96, 2688, 15\,840, 136\,800, \ldots\} \ .$$

The irreducible state counts for maximal entanglement are reduced by $(p+1)$, giving, for $n=2$,

$$p\left(p^2-1\right) = \{24, 336, 1320, 6840, \ldots\} \ .$$

For three qubits, there are $p^3\,(p^4-1)\,(p+1)$ (total) and $p^3\,(p^4-1)$ (irreducible) instances of pure maximally entangled states, while the general formula for 4-qubit states remains unclear.

Therefore, the ratio of maximally entangled to product states is

$$\frac{\textbf{Max entangled}}{\textbf{Product}} = \frac{p+1}{p\,(p-1)}, \ \frac{(p^2+1)\,(p+1)}{(p-1)^2} \ ,$$

for $n=2$ and 3, respectively.

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References

[1] Hanson A J, Ortiz G, Sabry A and Tai Y T 2013 *Journal of Physics A: Mathematical and Theoretical* **46** 185301 URL `http://stacks.iop.org/1751-8121/46/i=18/a=185301`

[2] Barnum H, Knill E, Ortiz G, Somma R and Viola L 2004 *Phys. Rev. Lett.* **92**(10) 107902 URL `http://link.aps.org/doi/10.1103/PhysRevLett.92.107902`

[3] Barnum H, Knill E, Ortiz G and Viola L 2003 *Phys. Rev. A* **68**(3) 032308 URL `http://link.aps.org/doi/10.1103/PhysRevA.68.032308`